

ON THE GEOMETRY OF CONFORMALLY STATIONARY LORENTZ SPACES

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ABSTRACT. In this paper we study several aspects of the geometry of conformally stationary Lorentz manifolds, and particularly of GRW spaces, due to the presence of a closed conformal vector field. More precisely, we begin by extending to these spaces a result of J. Simons on the minimality of cones in Euclidean space, and apply it to the construction of complete, noncompact maximal submanifolds of both de Sitter and anti-de Sitter spaces. Then we state and prove very general Bernstein-type theorems for spacelike hypersurfaces in conformally stationary Lorentz manifolds, one of which not assuming the hypersurface to be of constant mean curvature. Finally, we study the strong r -stability of spacelike hypersurfaces of constant r -th mean curvature in a conformally stationary Lorentz manifold of constant sectional curvature, extending previous results in the current literature.

1. INTRODUCTION

An important class of Lorentz manifolds is formed by the so-called stationary Lorentz manifolds. Following [17], Chapter 6, we say that a Lorentz manifold is stationary if there exists a one-parameter group of isometries whose orbits are timelike curves; for spacetimes, this group of isometries expresses time translation symmetry. From the mathematical viewpoint, a stationary Lorentz manifold is simply a Lorentz manifold furnished with a timelike Killing vector field, and a natural generalization is a *conformally stationary* Lorentz manifold, i.e., one furnished with a timelike *conformal* vector field. Our interest in conformally stationary Lorentz manifolds is due to the fact that, under an appropriate conformal change of metric, the conformal vector field turns into a Killing one, so that the new Lorentz manifold is now stationary.

Our aim in this work is to understand the geometry of immersed submanifolds of conformally stationary Lorentz manifolds furnished with a closed conformal vector field, and we do this by approaching three different kinds of problems: the construction of examples of maximal submanifolds, the obtainance of general Bernstein-type results and the derivation of suitable criteria for r -stability.

First of all (cf. Theorem 3.1), we extend a classical theorem of Simons [16] to conformally stationary Lorentz manifolds and apply it to build maximal Lorentz immersions whenever the ambient space either is of constant sectional curvature or has vanishing Ricci curvature in the direction of the conformal vector field; in

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particular, we provide a geometrical construction for maximal immersions into both the anti-de Sitter and de Sitter spaces (cf. Corollaries 3.3 and 3.4).

Related to Bernstein-type results, we study complete spacelike hypersurfaces, not necessarily of constant mean curvature, immersed into a conformally stationary Lorentz manifold of nonnegative Ricci curvature, furnished with homothetic non-parallel vector fields. By asking the second fundamental form of the hypersurface to be bounded and imposing a natural restriction on the projection of the conformal vector field of the ambient space, we classify such hypersurfaces in Theorems 4.1 and 4.3. In particular, we classify complete, finitely punctured spacelike radial graphs over \mathbb{R}^n or \mathbb{H}^n in \mathbb{L}^{n+1} , thus extending a classical result of J. Jellett [11] to the Lorentz context.

Finally, in the last section we derive in Theorem 5.7 a sufficient criterion for strong r -stability of spacelike hypersurfaces of constant r -th mean curvature. This result extends, to the class of conformally stationary Lorentz manifolds, previous ones obtained in [6] and [8] in the context of Generalized Robertson-Walker space-times.

2. CONFORMALLY STATIONARY LORENTZ MANIFOLDS

As in the previous section, let \overline{M}^{n+1} be a Lorentz manifold. We recall that a vector field V on \overline{M}^{n+1} is conformal if

$$(2.1) \quad \mathcal{L}_V \langle \cdot, \cdot \rangle = 2\psi \langle \cdot, \cdot \rangle$$

for some function $\psi \in C^\infty(\overline{M})$, where \mathcal{L} stands for the Lie derivative of the Lorentz metric of \overline{M} ; the function ψ is the conformal factor of V . Any Lorentz manifold \overline{M}^{n+1} possessing a globally defined, timelike conformal vector field is said to be conformally stationary.

Since $\mathcal{L}_V(X) = [V, X]$ for all $X \in \mathfrak{X}(\overline{M})$, the tensorial character of \mathcal{L}_V shows that $V \in \mathfrak{X}(\overline{M})$ is conformal if and only if

$$(2.2) \quad \langle \overline{\nabla}_X V, Y \rangle + \langle X, \overline{\nabla}_Y V \rangle = 2\psi \langle X, Y \rangle,$$

for all $X, Y \in \mathfrak{X}(\overline{M})$. In particular, V is Killing if and only if $\psi \equiv 0$, and (2.2) gives

$$\psi = \frac{1}{n+1} \operatorname{div}_{\overline{M}} V.$$

Suppose that our conformally stationary spacetime \overline{M}^{n+1} is endowed with a closed conformal timelike vector field V with conformal factor ψ , i.e., one for which

$$(2.3) \quad \overline{\nabla}_X V = \psi X$$

for all $X \in \mathfrak{X}(\overline{M})$. We recall that such a V is parallel if ψ vanishes identically and homothetic if ψ is constant¹.

If V has no singularities on an open set $\mathcal{U} \subset \overline{M}$, then the distribution V^\perp on \mathcal{U} of vector fields orthogonal to V is integrable, for if $X, Y \in V^\perp$, then

$$\langle [X, Y], V \rangle = \langle \overline{\nabla}_X Y - \overline{\nabla}_Y X, V \rangle = -\langle Y, \overline{\nabla}_X V \rangle + \langle X, \overline{\nabla}_Y V \rangle = 0.$$

We let Ξ be a leaf of V^\perp furnished with the induced metric, and D its Levi-Civita connection.

¹Here we diverge a little bit from other papers (e.g. [12]), where homothetic means just conformal with constant conformal factor. The reason is economy: to avoid constantly writing *closed* and *homothetic*.

From (2.3) we get

$$(2.4) \quad \bar{\nabla}\langle V, V \rangle = 2\psi V,$$

so that $\langle V, V \rangle$ is constant on connected leaves of V^\perp . Computing covariant derivatives in (2.4), we have

$$(\text{Hess}_{\bar{M}}\langle V, V \rangle)(X, Y) = 2X(\psi)\langle V, Y \rangle + 2\psi^2\langle X, Y \rangle.$$

However, since both $\text{Hess}_{\bar{M}}$ and the metric are symmetric tensors, we get

$$X(\psi)\langle V, Y \rangle = Y(\psi)\langle V, X \rangle$$

for all $X, Y \in \mathfrak{X}(\bar{M})$. Taking $Y = V$ we then arrive at

$$(2.5) \quad \bar{\nabla}\psi = \frac{V(\psi)}{\langle V, V \rangle}V = -\nu(\psi)\nu,$$

where $\nu = \frac{V}{\sqrt{-\langle V, V \rangle}}$. Hence, ψ is also constant on connected leaves of V^\perp . If Ξ is such a leaf and S_Ξ denotes its shape operator with respect to ν , we get

$$(2.6) \quad S_\Xi(X) = -\bar{\nabla}_X\nu = \psi X,$$

and hence Ξ is an umbilical hypersurface of \bar{M} .

Now we need the following

Lemma 2.1. *If η is another closed conformal vector field on \bar{M} and $U = \eta + \langle \eta, \nu \rangle \nu$, then U is closed conformal on Ξ , with conformal factor $\psi_U = \psi_\eta + \psi_\nu \langle \eta, \nu \rangle$.*

Proof. For $Z \in \mathfrak{X}(\Xi)$, it follows from $\langle Z, \nu \rangle = 0$ that

$$\begin{aligned} D_Z U &= (\bar{\nabla}_Z U)^\top = \bar{\nabla}_Z U + \langle \bar{\nabla}_Z U, \nu \rangle \nu \\ &= \bar{\nabla}_Z(\eta + \langle \eta, \nu \rangle \nu) + \langle \bar{\nabla}_Z(\eta + \langle \eta, \nu \rangle \nu), \nu \rangle \nu \\ &= \bar{\nabla}_Z \eta + Z\langle \eta, \nu \rangle \nu + \langle \eta, \nu \rangle \bar{\nabla}_Z \nu + \langle \bar{\nabla}_Z \eta, \nu \rangle \nu \\ &\quad + Z\langle \eta, \nu \rangle \langle \nu, \nu \rangle \nu + \langle \eta, \nu \rangle \langle \bar{\nabla}_Z \nu, \nu \rangle \nu \\ &= \bar{\nabla}_Z \eta + \langle \eta, \nu \rangle \bar{\nabla}_Z \nu + \langle \bar{\nabla}_Z \eta, \nu \rangle \nu \\ &= \psi_\eta Z + \langle \eta, \nu \rangle \psi_\nu Z + \langle \psi_\eta Z, \nu \rangle \nu \\ &= (\psi_\eta + \langle \eta, \nu \rangle \psi_\nu) Z. \end{aligned} \tag{2.7}$$

□

Example 2.2. Let \mathbb{L}^{n+1} be the $(n+1)$ -dimensional Lorentz space with its usual scalar product $\langle \cdot, \cdot \rangle$, with respect to the quadratic form $q(x) = \sum_{i=1}^n x_i^2 - x_{n+1}^2$. For $n > 2$, the n -dimensional de Sitter space is the hyperquadric

$$\mathbb{S}_1^n = \{x \in \mathbb{L}^{n+1}; \langle x, x \rangle = 1\},$$

and also the Lorentz simply-connected space form of constant sectional curvature identically equal to 1. The previous proposition teaches how to geometrically build closed conformal vector field on \mathbb{S}_1^n : since $\nu \in \mathfrak{X}(\mathbb{L}^{n+1})$ given by $\nu(x) = x$ is homothetic, choose a parallel $\eta \in \mathfrak{X}(\mathbb{L}^{n+1})$ and project it orthogonally onto the \mathbb{S}_1^n .

Example 2.3. Let R_2^{n+1} denote R^{n+1} furnished with the scalar product correspondent to the quadratic form $q(x) = \sum_{i=1}^{n-1} x_i^2 - x_n^2 - x_{n+1}^2$. For $n > 1$, the n -dimensional anti-de Sitter space is the hyperquadric

$$\mathbb{H}_1^n = \{x \in \mathbb{R}_2^{n+1}; \langle x, x \rangle = -1\},$$

and also the Lorentz simply-connected space form of constant sectional curvature identically equal to -1 . A construction similar to that of the previous example can obviously be made for \mathbb{H}_1^n .

Following [1], a particular class of conformally stationary spacetimes is that of Generalized Robertson-Walker spaces (GRW for short), namely, warped products $\overline{M}^{n+1} = -I \times_\phi F^n$, where $I \subset \mathbb{R}$ is an open interval with the metric $-dt^2$, F^n is an n -dimensional Riemannian manifold and $\phi : I \rightarrow \mathbb{R}$ is positive and smooth. For such a space, if $\pi_I : \overline{M}^{n+1} \rightarrow I$ is the canonical projection onto I , then the vector field

$$V = (\phi \circ \pi_I) \partial_t$$

is a conformal, timelike and closed, with conformal factor $\psi = \phi' \circ \pi_I$, where the prime denotes differentiation with respect to t . Moreover (cf. [13]), for $t_0 \in I$, the (spacelike) leaf $\Xi_{t_0}^n = \{t_0\} \times F^n$ is totally umbilical, with umbilicity factor $-\frac{\phi'(t_0)}{\phi(t_0)}$ with respect to the future-pointing unit normal vector field.

Remark 2.4. Conversely, let \overline{M} be a general conformally stationary Lorentz manifold with closed conformal vector field V . If $p \in \overline{M}$ and Ξ_p is the leaf of V^\perp passing through p , then we can find a neighborhood \mathcal{U}_p of p in Ξ_p and an open interval $I \subset \mathbb{R}$ containing 0 such that the flow Ψ of V is defined on \mathcal{U}_p for every $t \in I$. Besides, if \overline{M} is timelike geodesically complete, S. Montiel [13] proved that

$$\begin{aligned} \varphi : \mathbb{R} \times \Xi_p &\longrightarrow \overline{M} \\ (t, p) &\longmapsto \Psi(t, p) \end{aligned}$$

is a global parametrization on \overline{M} , such that \overline{M} is isometric to the GRW $-\mathbb{R} \times_\phi \Xi_p$, where $\phi(q) = t \Leftrightarrow \Psi(-t, q) \in \Xi_p$.

Let $x : M^n \rightarrow \overline{M}^{n+1}$ a spacelike immersion, that is, the induced metric via x on M^n is Riemannian. According to [1], if $\overline{M}^{n+1} = -I \times_\phi F^n$ is a GRW and $x : M^n \rightarrow \overline{M}^{n+1}$ is a complete spacelike hypersurface such that $\phi \circ \pi_I$ is bounded on M^n , then $\pi_F|_M : M^n \rightarrow F^n$ is necessarily a covering map. In particular, if M^n is closed, then F^n is automatically closed.

Also, recall (cf. Chapter 7 of [14]) that a GRW as above has constant sectional curvature c if and only if F has constant sectional curvature k and the warping function ϕ satisfies the ODE

$$(2.8) \quad \frac{\phi''}{\phi} = c = \frac{(\phi')^2 + k}{\phi^2}.$$

Example 2.5. The de Sitter space \mathbb{S}_1^{n+1} is an important particular example of GRW of constant sectional curvature 1. In fact, it follows from (2.8) and the classification of simply connected Lorentz space forms that

$$\mathbb{S}_1^{n+1} \simeq -\mathbb{R} \times_{\cosh t} \mathbb{S}^n,$$

where \mathbb{S}^n is the standard n -dimensional unit sphere in Euclidean space. Hence, the vector field $V = (\sinh t) \partial_t$ is a timelike closed conformal one. The equator of \mathbb{S}_1^{n+1} is the slice $\{0\} \times \mathbb{S}^n$, and the points $(t, p) \in \mathbb{S}_1^{n+1}$ with $t < 0$ (resp. $t > 0$) are said to form the chronological past (resp. future) of \mathbb{S}_1^{n+1} .

Example 2.6. Another important example is given by the anti-de Sitter space \mathbb{H}_1^{n+1} . Invoking once more (2.8) and the classification of simply connected Lorentz space forms, we get the isometry

$$\mathbb{H}_1^{n+1} \simeq -(0, \pi) \times_{\sin t} \mathbb{H}^n,$$

where \mathbb{H}^n is the n -dimensional hyperbolic space. Therefore, the vector field $V = (\sin t)\partial_t$ is timelike closed and conformal in \mathbb{H}_1^{n+1} .

3. MAXIMAL SUBMANIFOLDS OF CONFORMALLY STATIONARY SPACES

In this section we generalize to conformally stationary Lorentz manifolds a theorem of J. Simons [16], which shows how one can build isometric immersions with parallel mean curvature in \mathbb{R}^{n+k+1} from minimal immersions $\varphi : M^n \rightarrow \mathbb{S}^{n+k}$.

As in the previous section, let \overline{M}^{n+k+1} be an $(n+k+1)$ -dimensional conformally stationary Lorentz manifold, with closed conformal vector field V of conformal factor ψ . If $V \neq 0$ on \overline{M} , we saw that the orthogonal distribution V^\perp is integrable, with totally umbilical leaves. Therefore, if Ξ^{n+k} be such a leaf, then it is an umbilical spacelike hypersurface of \overline{M}^{n+k+1} and $\nu = \frac{V}{\sqrt{-\langle V, V \rangle}}$ is a global unit normal timelike vector field on it.

Let $\varphi : M^n \rightarrow \Xi^{n+k}$ be an isometric immersion, where M^n is a compact Riemannian manifold. If Ψ denotes the flow of V , the compactness of M^n guarantees the existence of $\epsilon > 0$ such that Ψ is defined on $(-\epsilon, \epsilon) \times \varphi(M)$, and the map

$$(3.1) \quad \begin{array}{ccc} \Phi : (-\epsilon, \epsilon) \times M^n & \longrightarrow & \overline{M}^{n+k+1} \\ (t, q) & \longmapsto & \Psi(t, \varphi(q)) \end{array}$$

is also an immersion. Furnishing $(-\epsilon, \epsilon) \times M^n$ with the metric induced by Φ , we turn it into a Lorentz manifold and Φ into an isometric immersion such that $\Phi|_{\{0\} \times M^n} = \varphi$.

Finally, letting $\text{Ric}_{\overline{M}}$ denote the field of self-adjoint operators associated to the Ricci tensor of \overline{M} , we get the above-mentioned generalization of Simons' result (see also [10] for the Riemannian case).

Theorem 3.1. *In the above notations, let $\psi \neq 0$ on $\varphi(M)$. If \overline{M} has constant sectional curvature or $\text{Ric}_{\overline{M}}(V) = 0$, then the following are equivalent:*

- (a) φ is maximal.
- (b) Φ is maximal.
- (c) Φ has parallel mean curvature.

Proof. Fix $p \in M$ and, on a neighborhood Ω of p in M , an orthonormal frame $\{e_1, \dots, e_n, \eta_1, \dots, \eta_k\}$ adapted to φ , such that $\{e_1, \dots, e_n\}$ is geodesic at p .

If $E_1, \dots, E_n, N_1, \dots, N_k$ are the vector fields on $\Psi((-\epsilon, \epsilon) \times \Omega)$ obtained from the e_i 's and η_j 's by parallel transport along the integral curves of V that intersect Ω , it follows that $\{E_1, \dots, E_n, \nu, N_1, \dots, N_k\}$ is an orthonormal frame on $\Psi((-\epsilon, \epsilon) \times \Omega)$, adapted to the immersion (3.1).

Let $\overline{\nabla}$ be the Levi-Civita connection of \overline{M} and \overline{H} the mean curvature vector of Φ . It follows from the closed conformal character of V that, on $\Phi((-\epsilon, \epsilon) \times \Omega)$,

$$(3.2) \quad \overline{H} = \frac{1}{n+1}(\overline{\nabla}_{E_i} E_i + \overline{\nabla}_\nu \nu)^\perp = \frac{1}{n+1}(\tilde{\nabla}_{E_i} E_i)^\perp,$$

where \perp denotes orthogonal projection on $T\Phi((-\epsilon, \epsilon) \times \Omega)^\perp$.

In order to compute $\bar{\nabla}_{E_i} E_i$ along the integral curve that passes through p , note that

$$(3.3) \quad \langle \bar{\nabla}_{E_i} E_i, V \rangle = -\langle E_i, \bar{\nabla}_{E_i} V \rangle = -n\psi.$$

Now, if \bar{R} stands for the curvature operator of \bar{M} , observe that

$$(3.4) \quad \begin{aligned} \frac{d}{dt} \langle \bar{\nabla}_{E_i} E_i, E_k \rangle &= \langle \bar{\nabla}_V \bar{\nabla}_{E_i} E_i, E_k \rangle \\ &= \langle \bar{R}(V, E_i) E_i, E_k \rangle + \langle \bar{\nabla}_{E_i} \bar{\nabla}_V E_i, E_k \rangle + \langle \bar{\nabla}_{[V, E_i]} E_i, E_k \rangle \\ &= \langle \text{Ric}_{\bar{M}}(V), E_k \rangle - \langle \bar{\nabla}_{\bar{\nabla}_{E_i} V} E_i, E_k \rangle \\ &= -\psi \langle \bar{\nabla}_{E_i} E_i, E_k \rangle. \end{aligned}$$

Note that, in the last equality, we used the fact that either \bar{M} has constant sectional curvature or $\text{Ric}_{\bar{M}}(V) = 0$ to conclude that $\langle \text{Ric}_{\bar{M}}(V), E_k \rangle = 0$.

Let D and ∇ respectively denote the Levi-Civita connections of Ξ^{n+k} and M^n . Since $\{e_1, \dots, e_n\}$ is geodesic at p (on M), it follows that

$$(3.5) \quad \langle \bar{\nabla}_{E_i} E_i, E_k \rangle_p = \langle D_{e_i} e_i, e_k \rangle_p = \langle (D_{e_i} e_i)^\perp + \nabla_{e_i} e_i, e_k \rangle_p = 0.$$

Therefore, solving the Cauchy problem formed by (3.4) and (3.5), we get

$$(3.6) \quad \langle \bar{\nabla}_{E_i} E_i, E_k \rangle_{\Psi(t,p)} = 0, \quad \forall |t| < \epsilon.$$

Analogously to (3.4), we get

$$(3.7) \quad \frac{d}{dt} \langle \bar{\nabla}_{E_i} E_i, N_\beta \rangle = -\psi \langle \bar{\nabla}_{E_i} E_i, N_\beta \rangle.$$

On the other hand, letting $A_\beta : T_p M \rightarrow T_p M$ denote the shape operator of φ in the direction of η_β and writing $A_\beta e_i = h_{ij}^\beta e_j$, we have

$$(3.8) \quad \langle \bar{\nabla}_{E_i} E_i, N_\beta \rangle_p = \langle D_{e_i} e_i, \eta_\beta \rangle_p = \langle A_\beta e_i, e_i \rangle_p = h_{ii}^\beta.$$

Solving the Cauchy problem formed by (3.7) and (3.8), we get

$$(3.9) \quad \langle \bar{\nabla}_{E_i} E_i, N_\beta \rangle_{\Psi(t,p)} = h_{ii}^\beta \exp \left(- \int_0^t \psi(s) ds \right).$$

It finally follows from (3.3), (3.6) and (3.9) that, at the point (t, p) ,

$$\begin{aligned} \bar{\nabla}_{E_i} E_i &= \langle \bar{\nabla}_{E_i} E_i, E_k \rangle E_k - \langle \bar{\nabla}_{E_i} E_i, \nu \rangle \nu + \langle \bar{\nabla}_{E_i} E_i, N_\beta \rangle N_\beta \\ &= \frac{n\psi}{\langle V, V \rangle} V + \exp \left(- \int_0^t \psi(s) ds \right) h_{ii}^\beta N_\beta. \end{aligned}$$

Therefore, (3.2) gives us

$$(3.10) \quad \bar{H} = \frac{1}{n+1} \exp \left(- \int_0^t \psi(s) ds \right) h_{ii}^\beta N_\beta.$$

Let us finally establish the equivalence of (a), (b) and (c), observing that (b) \Rightarrow (c) is always true.

(a) \Rightarrow (b): if φ is maximal, we have $h_{ii}^\beta = 0$ for all $1 \leq \beta \leq k$, and it follows from (3.10) that $\bar{H} = 0$.

(c) \Rightarrow (a): if $\overline{\nabla}^\perp \overline{H} = 0$, then, along the integral curve of V that passes through p , the parallelism of the N_β gives

$$0 = \overline{\nabla}_V^\perp \overline{H} = \left(\frac{D\overline{H}}{dt} \right)^\perp = -\frac{\psi(t)}{n+1} \exp \left(-\int_0^t \psi(s) ds \right) h_{ii}^\beta N_\beta.$$

However, since $\psi \neq 0$ on $\varphi(M)$, it follows from the above equality that $h_{ii}^\beta = 0$ at p for all $1 \leq \beta \leq k$, so that φ is maximal at p . \square

The following corollaries are immediate.

Corollary 3.2. *Let $I \subset (0, +\infty)$ and $t_0 \in I$. In the GRW space $-I \times_t \mathbb{F}^{n+k}$, if $\varphi : M^n \rightarrow \{t_0\} \times F^{n+k}$ is an isometric immersion and Φ is the canonical immersion of $-I \times_t M^n$ into $-I \times_t F^{n+k}$, then φ is maximal if and only if Φ is maximal.*

Proof. Since $V = t\partial_t$, Corollary 7.43 of [14], $\text{Ric}_{\overline{M}}(V) = 0$ in this case. \square

Corollary 3.3. *Let $\varphi : M^n \rightarrow \mathbb{S}^{n+k}$ be an n -dimensional submanifold of some round sphere \mathbb{S}^{n+k} of the $(n+k+1)$ -dimensional de Sitter space \mathbb{S}_1^{n+k+1} . If $\varphi(M)$ is contained in the chronological past (resp. future) of \mathbb{S}_1^{n+k+1} , then M^n is maximal in \mathbb{S}^{n+k} if and only if the union of the segments of the integral curves of ∂_t contained in the chronological past (resp. future) of \mathbb{S}_1^{n+k+1} and passing through points of $\varphi(M)$ is maximal in \mathbb{S}_1^{n+k+1} .*

Corollary 3.4. *Let $\varphi : M^n \rightarrow \mathbb{H}^{n+k}$ be an n -dimensional submanifold of some hyperbolic space \mathbb{H}^{n+k} of the $(n+k+1)$ -dimensional anti-de Sitter space \mathbb{H}_1^{n+k+1} . Then M^n is maximal in \mathbb{H}^{n+k} if and only if the union of the segments of the integral curves of ∂_t that pass through points of $\varphi(M)$ is maximal in \mathbb{H}_1^{n+k+1} .*

4. BERNSTEIN-TYPE THEOREMS

We continue to employ the notations of the previous sections, i.e., \overline{M} is conformally stationary with closed conformal vector field V . However, we let ψ_V be the conformal factor of V .

From now on, we let $x : M^n \rightarrow \overline{M}^{n+1}$ be a connected, complete, oriented spacelike hypersurface, N be a unit normal vector field which orients M and has the same time-orientation of V , and A and H be respectively the shape operator and the mean curvature of M with respect to N .

If $f_V : M \rightarrow \mathbb{R}$ is given by $f_V = \langle V, N \rangle$ then f_V is negative on M . On the other hand, standard computations (cf. [6]) give

$$(4.1) \quad \nabla f_V = -A(V^\top)$$

and

$$(4.2) \quad \Delta f_V = nV^\top(H) + \{\text{Ric}_{\overline{M}}(N, N) + |A|^2\} f_V + n\{H\psi + N(\psi)\},$$

where $(\)^\top$ stands for orthogonal projection onto M .

If W is another closed conformal vector field on \overline{M} , with conformal factor ψ_W , and $g : M \rightarrow \mathbb{R}$ is given by $g = \langle V, W \rangle$, then another standard computation gives

$$(4.3) \quad \nabla g = \psi_V W^\top + \psi_W V^\top.$$

and

$$(4.4) \quad \Delta g = W^\top(\psi_V) + V^\top(\psi_W) + nH(\psi_V f_W + \psi_W f_V) + 2n\psi_V \psi_W.$$

We are now in position to state and prove the following Bernstein-type general theorem for spacelike hypersurfaces. Observe that we do not require the hypersurface in question to be of constant mean curvature. In what follows, we let $\mathcal{L}^1(M)$ be the space of Lebesgue integrable functions on M .

Theorem 4.1. *Let \overline{M}^{n+1} have nonnegative Ricci curvature, V and W be respectively a parallel and a homothetic nonparallel vector field on \overline{M}^{n+1} , and $x : M^n \rightarrow \overline{M}^{n+1}$ be as above. If $|A|$ is bounded, $|V^\top|$ is integrable and H doesn't change sign on M , then:*

- (a) *M is totally geodesic and the Ricci curvature of \overline{M} in the direction of N vanishes identically.*
- (b) *If M is noncompact and Ric_M is also nonnegative, then $x(M)$ is contained in a leaf of V^\perp .*

Proof.

(a) Since V is parallel and W is homothetic and nonparallel, it follows from (4.1), (4.3), (4.2) and (4.4) that $\nabla f_V = -A(V^\top)$, $\nabla g = \psi_W V^\top$,

$$(4.5) \quad \Delta f_V = nV^\top(H) + (\text{Ric}_{\overline{M}}(N, N) + |A|^2)f_V$$

and

$$\Delta g = nH\psi_W f_V,$$

with ψ_W being a nonzero constant. Therefore, the assumption $|V^\top| \in \mathcal{L}^1(M)$ gives $|\nabla g| \in \mathcal{L}^1(M)$, and the assumption on H , together with the fact that $|f_V| > 0$ on M , assures that Δg is either nonnegative or nonpositive on M . Therefore, the Corollary on page 660 of [19] gives $\Delta g = 0$ on M , and hence $H = 0$ on M .

We now look at (4.5), which resumes to

$$\Delta f_V = (\text{Ric}_{\overline{M}}(N, N) + |A|^2)f_V,$$

and hence doesn't change sign on M too. We also note that the boundedness of $|A|$ on M gives

$$|\nabla f_V| \leq |A||V^\top| \in \mathcal{L}^1(M).$$

Appealing again to the Corollary on page 660 of [19], we get $\Delta f_V = 0$ on M , so that

$$\text{Ric}_{\overline{M}}(N, N) + |A|^2 = 0$$

on M . Since $\text{Ric}_{\overline{M}}(N, N) \geq 0$, then we get $\text{Ric}_{\overline{M}}(N, N) = 0$ and $A = 0$ on M , i.e., M is totally geodesic.

(b) $A = 0$ on M gives $\nabla f_V = 0$ on M , so that $f_V = \langle V, N \rangle$ is constant and nonzero on M . However, $|V|^2$ is constant on \overline{M} (since V is parallel) and

$$|V^\top|^2 = |V|^2 + \langle V, N \rangle^2,$$

so that $|V^\top|$ is also constant on M . Therefore,

$$+\infty > \int_M |V^\top| dM = |V^\top| \text{Vol}(M).$$

But since M is noncompact and has nonnegative Ricci curvature, another theorem of Yau (Theorem 7 of [19]) gives $\text{Vol}(M) = +\infty$, and hence the only possibility is $|V^\top| = 0$. Therefore, Cauchy-Schwarz inequality gives that V is parallel to N , and $x(M)$ is contained in a leaf of V^\perp . \square

For the next result we need a small refinement of the analytical tool of Yau's result used in the above proof. We quote it below, referring the reader to [10] for a proof.

Lemma 4.2. *Let $X \in \mathfrak{X}(M)$ be such that $\operatorname{div}_M X$ doesn't change sign on M . If $|X| \in \mathcal{L}^1(M)$, then $\operatorname{div} X = 0$ on M .*

Theorem 4.3. *Let \overline{M} have nonnegative Ricci curvature, V be a homothetic vector field on \overline{M}^{n+1} , and $x : M^n \rightarrow \overline{M}^{n+1}$ be as before. If $|A|$ is bounded, $|V^\top|$ is integrable and H is constant on M , then M is totally umbilical and the Ricci curvature of \overline{M} in the direction of N vanishes identically.*

Proof. Since H is constant on M and ψ_V is constant on \overline{M} , (4.2) reduces to

$$\Delta f_V = \{\operatorname{Ric}_{\overline{M}}(N, N) + |A|^2\} f_V + nH\psi_V.$$

Letting $\{e_1, \dots, e_n\}$ be a moving frame on M , we have

$$\begin{aligned} \operatorname{div}_M(V^\top) &= \langle \overline{\nabla}_{e_k}(V + f_V N), e_k \rangle \\ (4.6) \quad &= n\psi_V - f_V \langle A(e_k), e_k \rangle \\ &= n\psi_V + nHf_V, \end{aligned}$$

so that

$$\begin{aligned} \operatorname{div}_M(\nabla f_V - HV^\top) &= \Delta f_V - nH\psi_V - nH^2 f_V \\ (4.7) \quad &= \{\operatorname{Ric}_{\overline{M}}(N, N) + |A|^2 - nH^2\} f_V. \end{aligned}$$

Since $|f_V| > 0$ on M , $\operatorname{Ric}_{\overline{M}}(N, N) \geq 0$ and $|A|^2 \geq nH^2$ by Cauchy-Schwarz inequality (with equality if and only if M is totally umbilical), this last expression does not change sign on M . Now observe that

$$|\nabla f_V - HV^\top| = |-AV^\top - HV^\top| \leq (|A| + H)|V^\top| \in \mathcal{L}^1(M),$$

so that the previous lemma gives $\operatorname{div}_M(\nabla f_V - HV^\top) = 0$ on M . Back to (4.7), we then get $\operatorname{Ric}_{\overline{M}}(N, N) = 0$ and $|A|^2 - nH^2 = 0$, and hence M is totally umbilical. \square

The previous result yields the following corollary on GRW spaces.

Corollary 4.4. *Let $I \subset (0, +\infty)$ be an open interval, F^n be an n -dimensional, complete oriented Riemannian manifold of nonnegative Ricci curvature, $\overline{M}^{n+1} = -I \times_t F^n$ and $x : M^n \rightarrow \overline{M}^{n+1}$ as before. If $|A|$ is bounded, $|(t\partial_t)^\top| \in \mathcal{L}^1(M)$ and H is constant on M , then M is totally umbilical and the Ricci curvature of \overline{M} in the direction of N vanishes identically. In particular, if F is closed and has positive Ricci curvature everywhere, then $x(M) \subset \{t_0\} \times F$, for some $t_0 \in I$.*

Proof. The first part follows from the theorem. To the second one, if F is closed and has positive Ricci curvature everywhere, then, according to the previous result, the only possible direction for N is that of $t\partial_t$. But if N is parallel to ∂_t , then the connectedness of M guarantees that $x(M)$ cannot jump from one leaf $\{t_0\} \times F$ to another. \square

In what follows, we let $\mathbb{R}^n = \{x \in \mathbb{L}^{n+1}; x_{n+1} = 0\}$ and $\mathbb{H}^n = \{x \in \mathbb{L}^{n+1}; \langle x, x \rangle = -1, x_{n+1} > 0\}$. As a special case of the previous corollary, we get

Corollary 4.5. *Let $x : M^n \rightarrow \mathbb{L}^{n+1}$ be an embedding, such that $x(M)$ is a complete spacelike radial graph over either \mathbb{R}^n or \mathbb{H}^n , minus k points. If $|A|$ is bounded, H is constant and $p \mapsto |x(p)^\top|$ is integrable on M , then $k = 0$ and $x(M)$ is either a spacelike hyperplane or a translation of \mathbb{H}^n .*

Remark 4.6. The class of examples of Corollary 3.3 of [9] shows that the hypothesis on the integrability of $p \mapsto |x(p)^\top|$ is really necessary.

5. r -STABILITY OF SPACELIKE HYPERSURFACES

For the time being, let \overline{M}^{n+1} denote a time-oriented Lorentz manifold (i.e., not necessarily conformally stationary) with Lorentz metric $\overline{g} = \langle \cdot, \cdot \rangle$, volume element $d\overline{M}$ and semi-Riemannian connection $\overline{\nabla}$. We consider spacelike hypersurfaces $x : M^n \rightarrow \overline{M}^{n+1}$, namely, isometric immersions from a connected, n -dimensional orientable Riemannian manifold M^n into \overline{M} . We let ∇ denote the Levi-Civita connection of M^n .

If \overline{M} is time-orientable and $x : M^n \rightarrow \overline{M}^{n+1}$ is a spacelike hypersurface, then M^n is orientable (cf. [14]) and one can choose a globally defined unit normal vector field N on M^n having the same time-orientation of \overline{M} ; such an N is said to be a *future-pointing Gauss map* of M^n . If we let A denote the shape operator of x with respect to N , then A restricts to a self-adjoint linear map $A_p : T_p M \rightarrow T_p M$ at each $p \in M^n$.

For $1 \leq r \leq n$, let $S_r(p)$ denote the r -th elementary symmetric function on the eigenvalues of A_p , so that one gets n smooth functions $S_r : M^n \rightarrow \mathbb{R}$ for which

$$\det(t\text{Id} - A) = (-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by definition. For fixed $p \in M^n$, the spectral theorem allows us to choose on $T_p M$ an orthonormal basis $\{e_1, \dots, e_n\}$ of eigenvectors of A_p , with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$, respectively. One thus immediately sees that

$$S_r = \sigma_r(\lambda_1, \dots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[X_1, \dots, X_n]$ is the r -th elementary symmetric polynomial on the indeterminates X_1, \dots, X_n .

For $1 \leq r \leq n$, one defines the r -th mean curvature H_r of x by

$$\binom{n}{r} H_r = (-1)^r S_r = \sigma_r(-\lambda_1, \dots, -\lambda_n).$$

One also let the r -th Newton transformation P_r on M^n be given by setting $P_0 = \text{Id}$ and, for $1 \leq r \leq n$, via the recurrence relation

$$(5.1) \quad P_r = (-1)^r S_r \text{Id} + A P_{r-1}.$$

A trivial induction shows that

$$P_r = (-1)^r (S_r \text{Id} - S_{r-1} A + S_{r-2} A^2 - \dots + (-1)^r A^r),$$

so that Cayley-Hamilton theorem gives $P_n = 0$. Moreover, since P_r is a polynomial in A for every $1 \leq r \leq n$, it is also self-adjoint and commutes with A . Therefore, all bases of $T_p M$ diagonalizing A at $p \in M^n$ also diagonalize all of the P_r at p . If $\{e_1, \dots, e_n\}$ is such a basis and A_i denotes the restriction of A to $\langle e_i \rangle^\perp \subset T_p \Sigma$, it is easy to see that

$$\det(t\text{Id} - A_i) = (-1)^k S_k(A_i) t^{n-1-k},$$

where

$$S_k(A_i) = \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_1, \dots, j_k \neq i}} \lambda_{j_1} \cdots \lambda_{j_k}.$$

With the above notations, it is also immediate to check that $P_r e_i = (-1)^r S_r(A_i) e_i$, and it is a standard fact that

- (i) $\text{tr}(P_r) = (-1)^r (n - r) S_r = b_r H_r$;
- (ii) $\text{tr}(A P_r) = (-1)^r (r + 1) S_{r+1} = -b_r H_{r+1}$;
- (iii) $\text{tr}(A^2 P_r) = (-1)^r (S_1 S_{r+1} - (r + 2) S_{r+2})$,

where $b_r = (n - r) \binom{n}{r}$.

Associated to each Newton transformation P_r one has the second order linear differential operator $L_r : \mathcal{D}(M) \rightarrow \mathcal{D}(M)$ given by

$$L_r(f) = \text{tr}(P_r \text{Hess } f).$$

In particular, $L_0 = \Delta$, the Laplace operator on smooth functions on M . If \overline{M}^{n+1} is of constant sectional curvature, H. Rosenberg [15] proved that

$$L_r(f) = \text{div}(P_r \nabla f).$$

A variation of x is a smooth mapping

$$X : M^n \times (-\epsilon, \epsilon) \rightarrow \overline{M}^{n+1}$$

satisfying the following conditions:

- (i) For $t \in (-\epsilon, \epsilon)$, the map $X_t : M^n \rightarrow \overline{M}^{n+1}$ given by $X_t(p) = X(t, p)$ is a spacelike immersion such that $X_0 = x$.
- (ii) $X_t|_{\partial M} = x|_{\partial M}$, for all $t \in (-\epsilon, \epsilon)$.

The variational field associated to the variation X is the vector field $\left. \frac{\partial X}{\partial t} \right|_{t=0}$. Letting N_t denote the unit normal vector field along X_t and $f = -\langle \frac{\partial X}{\partial t}, N_t \rangle$, we get

$$(5.2) \quad \frac{\partial X}{\partial t} = f N_t + \left(\frac{\partial X}{\partial t} \right)^\top,$$

where \top stands for tangential components.

Following [4], we set the balance of volume of the variation X as the function $\mathcal{V} : (-\epsilon, \epsilon) \rightarrow \mathbb{R}$ given by

$$\mathcal{V}(t) = \int_{M \times [0, t]} X^*(d\overline{M}),$$

and we say that X is volume-preserving if \mathcal{V} is constant.

Letting dM_t denote the volume element of the metric induced on M by X_t , we recall the following standard result (cf. [18]).

Lemma 5.1. *Let \overline{M}^{n+1} be a time-oriented Lorentz manifold and $x : M^n \rightarrow \overline{M}^{n+1}$ a closed spacelike hypersurface. If $X : M^n \times (-\epsilon, \epsilon) \rightarrow \overline{M}^{n+1}$ is a variation of x , then*

$$\frac{d\mathcal{V}}{dt} = \int_M f dM_t.$$

In particular, X is volume-preserving if and only if $\int_M f dM_t = 0$ for all t .

We remark that Lemma 2.2 of [4] remains valid in the Lorentz context, i.e., if $f_0 : M \rightarrow \mathbb{R}$ is a smooth function such that $\int_M f_0 dM = 0$, then there exists a volume-preserving variation of M whose variational field is $f_0 N$.

According to [5], we define the r -area functional $\mathcal{A}_r : (-\epsilon, \epsilon) \rightarrow \mathbb{R}$ associated to the variation X by

$$\mathcal{A}_r(t) = \int_M F_r(S_1, S_2, \dots, S_r) dM_t,$$

where $S_r = S_r(t)$ and F_r is recursively defined by setting $F_0 = 1$, $F_1 = -S_1$ and, for $2 \leq r \leq n-1$,

$$F_r = (-1)^r S_r - \frac{c(n-r+1)}{r-1} F_{r-2}.$$

In particular, if $r = 0$, then \mathcal{A}_0 is the classical area functional.

The Lorentz analogue of Proposition 4.1 of [5] is stated in the following Lemma (for another proof, see Lemma 2.2 of [8]).

Lemma 5.2. *If $x : M^n \rightarrow \overline{M}_c^{n+1}$ is a closed spacelike hypersurface of the time-oriented Lorentz manifold \overline{M}_c^{n+1} of constant sectional curvature c , and $X : M^n \times (-\epsilon, \epsilon) \rightarrow \overline{M}_c^{n+1}$ a variation of x , then*

$$(5.3) \quad \frac{\partial S_{r+1}}{\partial t} = (-1)^{r+1} [L_r f + \text{ctr}(P_r) f - \text{tr}(A^2 P_r) f] + \left\langle \left(\frac{\partial X}{\partial t} \right)^\top, \nabla S_{r+1} \right\rangle.$$

As in [5], the previous lemma allows us to compute the first variation of the r -area functional, according to the following

Proposition 5.3. *Under the hypotheses of Lemma 5.2, we have*

$$(5.4) \quad \mathcal{A}'_r(t) = \int_M [(-1)^{r+1}(r+1)S_{r+1} + c_r] f dM_t,$$

where $c_r = 0$ if r is even and $c_r = -\frac{n(n-2)(n-4)\dots(n-r+1)}{(r-1)(r-3)\dots 2}(-c)^{(r+1)/2}$ if r is odd.

In order to characterize spacelike immersions of constant $(r+1)$ -th mean curvature, let λ be a real constant and $\mathcal{J}_r : (-\epsilon, \epsilon) \rightarrow \mathbb{R}$ be the Jacobi functional associated to X , i.e.,

$$\mathcal{J}_r(t) = \mathcal{A}_r(t) - \lambda \mathcal{V}(t).$$

As an immediate consequence of (5.4) we get

$$\mathcal{J}'_r(t) = \int_M [b_r H_{r+1} + c_r - \lambda] f dM_t,$$

where $b_r = (r+1)\binom{n}{r+1}$. Therefore, if we choose $\lambda = c_r + b_r \overline{H}_{r+1}(0)$, where

$$\overline{H}_{r+1}(0) = \frac{1}{\mathcal{A}_0(0)} \int_M H_{r+1}(0) dM$$

is the mean of the $(r+1)$ -th curvature $H_{r+1}(0)$ of M , we arrive at

$$\mathcal{J}'_r(t) = b_r \int_M [H_{r+1} - \overline{H}_{r+1}(0)] f dM_t.$$

Hence, a standard argument (cf. [3]) shows that M is a critical point of \mathcal{J}_r for all variations of x if and only if M has constant $(r+1)$ -th mean curvature.

As in [5], we wish to study spacelike immersions $x : M^n \rightarrow \overline{M}^{n+1}$ that maximize \mathcal{J}_r for all variations X of x . The above discussion shows that M must have constant

$(r+1)$ -th mean curvature and, for such an M , leads us naturally to compute the second variation of \mathcal{J}_r . This, in turn, motivates the following

Definition 5.4. Let \overline{M}_c^{n+1} be a Lorentz manifold of constant sectional curvature c , and $x : M^n \rightarrow \overline{M}_c^{n+1}$ be a closed spacelike hypersurface having constant $(r+1)$ -th mean curvature. We say that x is strongly r -stable if, for every smooth function $f : M \rightarrow \mathbb{R}$ one has $\mathcal{J}_r''(0) \leq 0$.

The sought formula for the second variation of \mathcal{J}_r is another straightforward consequence of Proposition 5.3.

Proposition 5.5. *Let $x : M^n \rightarrow \overline{M}_c^{n+1}$ be a closed spacelike hypersurface of constant $(r+1)$ -mean curvature H_{r+1} . If $X : M^n \times (-\epsilon, \epsilon) \rightarrow \overline{M}_c^{n+1}$ is a variation of x , then*

$$(5.5) \quad \mathcal{J}_r''(0) = (r+1) \int_M [L_r(f) + \text{ctr}(P_r)f - \text{tr}(A^2 P_r)f] f dM.$$

Back to the conformally stationary setting, in what follows we need a formula first derived in [2]. As stated below, it is the Lorentz version of the one stated and proved in [7].

Lemma 5.6. *Let \overline{M}_c^{n+1} be a conformally stationary Lorentz manifold having constant sectional curvature c and conformal vector field V . Let also $x : M^n \rightarrow \overline{M}_c^{n+1}$ be a spacelike hypersurface and N a future-pointing, unit normal vector field globally defined on M^n . If $\eta = \langle V, N \rangle$, then*

$$(5.6) \quad \begin{aligned} L_r \eta &= \text{tr}(A^2 P_r) \eta - c \text{tr}(P_r) \eta - b_r H_r N(\psi) \\ &\quad + b_r H_{r+1} \psi + \frac{b_r}{r+1} \langle V, \nabla H_{r+1} \rangle, \end{aligned}$$

where $\psi : \overline{M}^{n+1} \rightarrow \mathbb{R}$ is the conformal factor of V , H_j is the j -th mean curvature of x and ∇H_j stands for the gradient of H_j on M .

We are now in position to state and prove the following

Theorem 5.7. *Let \overline{M}_c^{n+1} be a timelike geodesically complete conformally stationary Lorentz manifold of constant sectional curvature c having a closed conformal timelike vector field V , and let $x : M^n \rightarrow \overline{M}_c^{n+1}$ be a closed, strongly r -stable spacelike hypersurface. Suppose that the conformal factor ψ of V satisfies the condition*

$$\frac{H_r}{\sqrt{-\langle V, V \rangle}} \frac{\partial \psi}{\partial t} \geq \max\{H_{r+1} \psi, 0\},$$

where $t \in \mathbb{R}$ denotes the real parameter of the flow of V . If the set where $\psi = 0$ has empty interior in M , then M^n is either r -maximal or a leaf of the foliation V^\perp .

Proof. Since M^n is strongly r -stable, it follows from (5.5) that

$$(5.7) \quad (r+1) \int_M [L_r(f) + \text{ctr}(P_r)f - \text{tr}(A^2 P_r)f] f dM \leq 0,$$

for all smooth $f : M \rightarrow \mathbb{R}$. In particular, since H_{r+1} is constant on M , taking $f = \eta = \langle V, N \rangle$ in (5.6), we get

$$L_r \eta + c \text{tr}(P_r) \eta - \text{tr}(A^2 P_r) \eta = -b_r H_r N(\psi) + b_r H_{r+1} \psi,$$

so that (5.7) gives

$$(5.8) \quad \int_M [-H_r N(\psi) + H_{r+1} \psi] \langle V, N \rangle dM \leq 0.$$

However, it follows from (2.5) that

$$N(\psi) = \langle N, \bar{\nabla} \psi \rangle = -\nu(\psi) \langle \nu, N \rangle = \frac{\partial \psi}{\partial t} \frac{\cosh \theta}{\sqrt{-\langle V, V \rangle}},$$

where θ is the hyperbolic angle between V and N . Substituting the above into (5.8), we finally arrive at

$$\int_M \left[H_r \frac{\partial \psi}{\partial t} \frac{\cosh \theta}{\sqrt{-\langle V, V \rangle}} - H_{r+1} \psi \right] \frac{\cosh \theta}{\sqrt{-\langle V, V \rangle}} dM \leq 0.$$

Arguing as in the end of the proof of Theorem 1.1 of [6], we get

$$H_r \frac{\partial \psi}{\partial t} (\cosh \theta - 1) = 0 \quad \text{and} \quad \frac{H_r}{\sqrt{-\langle V, V \rangle}} \frac{\partial \psi}{\partial t} = H_{r+1} \psi$$

on M . But since H_{r+1} is constant on M , either M is r -maximal or $H_{r+1} \neq 0$ on M . If this last case happens, the condition on the zero set of ψ on M , together with the above, gives $H_r \frac{\partial \psi}{\partial t} \neq 0$ on a dense subset of M , and hence $\cosh \theta = 1$ on this set. By continuity, $\cosh \theta = 1$ on M , so that M is a leaf of the foliation ν^\perp . \square

The following corollary is immediate.

Corollary 5.8. *Let $x : M^n \rightarrow \mathbb{S}_1^{n+1}$ be a closed, strongly r -stable spacelike hypersurface, such that the set of points in which M^n intersects the equator of \mathbb{S}_1^{n+1} has empty interior in M . If*

$$H_r \geq \max\{(\sinh t)H_{r+1}, 0\},$$

then either M^n is r -maximal or an umbilical round sphere.

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